

## 59th Annual UNSW School Mathematics Competition: Competition Problems and Solutions

Solutions by Denis Potapov<sup>1</sup>

### A Junior Division – Problems

#### Problem A1:

Three grasshoppers play the following leapfrog game. They start off at three vertices of a square. At each step, a grasshopper leaps over another one and lands at the point symmetric to the point where it was. That is, if a grasshopper at a point  $P$  leaps over a grasshopper at a point  $Q$ , then it lands at a point  $R$  where  $Q$  is the midpoint of  $PR$ . Is it possible for one of them to reach the fourth vertex of the square?

#### Problem A2:

A group of Pokémon GO players caught together 2021 pokemons. It turned out that every one caught at least one pokemon; and no two players caught the same numbers of pokemons. What is the maximum possible number of members in the group.

#### Problem A3:

Two players pick petals one after another off a daisy flower with 17 petals. On each move, a player is allowed to pick either one petal or two adjacent petals. The player who picks the last petal wins. Find the strategy which ensures that one player wins regardless of the other player's moves. Determine the winning player.

#### Problem A4:

The Fibonacci sequence is  $1, 1, 2, 3, 5, 8, 13, \dots$ ; or, formally, the sequence  $F = \{f_n\}_{n=1}^{\infty}$  of integers which is defined by the following recurring relation:

$$f_1 = 1, \quad f_2 = 1 \quad \text{and} \quad f_n = f_{n-1} + f_{n-2}, \quad n \geq 3.$$

Find all positive integers  $n$  such that  $f_n$  is divisible by 133.

#### Problem A5:

In a community of one hundred people, every one knows exactly three other people. On 1 Jan 2021, a member of the community arrived back carrying a virus and passed it on to the three of his acquaintances. In turn, on 2 Jan 2021, those three passed on the virus to each of their acquaintances not already infected and so on. It is known that, once person is infected, they remain infected; and the community was virus-free before 1 Jan 2021.

Is there a scenario such that everyone carries the virus on 14 Mar 2021 yet there is a person not carrying the virus on 5 Mar 2021?

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## B Senior Division – Problems

### Problem B1:

An ant crawls on a table with constant speed in one direction, then every 15 minutes changes direction by turning  $90^\circ$ . Prove that the ant can only return to the original position after a whole number of hours have elapsed.

### Problem B2:

The Fibonacci sequence is  $1, 1, 2, 3, 5, 8, 13, \dots$ ; or, formally, the sequence  $F = \{f_n\}_{n=1}^\infty$  of integers which is defined by the following recurring relation

$$f_1 = 1, \quad f_2 = 1 \quad \text{and} \quad f_n = f_{n-1} + f_{n-2}, \quad n \geq 3.$$

Find the remainder when  $f_{2021}$  is divided by 19.

### Problem B3:

An island inhabitants design an emergency response service based on helicopters. Their island is a disc of radius 100km and they plan to purchase helicopters capable of flying 300km/h. Find the minimum number helicopters needed for the emergency service in order to be able to reach every point of the island within 10min. Assume that helicopter's takeoff and landing times are negligible.

### Problem B4:

Prove that the number  $4n^4 + m^4$  is not prime if  $m, n$  are positive integers and  $m \neq n$ .

### Problem B5:

A company has one Director, ten Senior Managers, one hundred Site Supervisors, and one thousand Field Workers. Decisions are made according to votes, and according to Rulings by the Fair Work Commission. The Director holds 1000 votes, each Senior Manager holds 100 votes, each Site Supervisor holds 10 votes and each Field Worker holds one vote. The Fair Work Commission takes one month to make a Ruling.

The company is to distribute \$100,000 between its employees according to the following process: First, the Director proposes a distribution plan. If this has majority support in a vote, then the money is distributed according to this plan. Otherwise, the Fair Work Commission rules that the Director is no longer entitled to vote or share in the money.

Next, the Senior Managers propose a distribution plan that has their unanimous support. If this has majority support in a vote, then the money is distributed according to this plan. Otherwise, the Fair Work Commission rules that the Senior Managers are no longer entitled to vote or share in the money.

The Supervisors then propose a distribution plan that has their unanimous support. If this has majority support in a vote, then the money is distributed according to this plan. Otherwise, the Fair Work Commission rules that the Supervisors are not entitled to any of the money, and the money is distributed evenly to the Field Workers. All employees wish to maximise their share in the earnings, and, in the case of equal outcomes, they wish to minimise the time to achieve the outcome.

Find the Director's plan which will not be voted out, and which will ensure the largest possible earnings share for the Director.

## A Junior Division – Solutions

*Solution A1.*

Answer: No.

Assume that a coordinate system is given such that the original positions are  $(0, 0)$ ,  $(0, 1)$  and  $(1, 0)$ . If two grasshoppers have the coordinates  $(x, y)$  and  $(a, b)$  and the one at  $(x, y)$  takes the leap, then it lands at  $(2a - x, 2b - y)$ .

We infer that (a) the grasshoppers will always stay on integer coordinates; and that (b) if  $x + y$  is odd at starting position, then  $x + y$  is odd at the landing position.

Hence, no grasshopper will land at even position  $(1, 1)$ .  $\square$

*Solution A2.*

Answer: 63.

Let the number of players be  $n$  and let  $a_1$  the number of pokemons caught by the 1st player,  $a_2$  the number of pokemons caught by the 2nd player and so on. Assume that

$$a_1 < a_2 < \cdots < a_n.$$

It is known that  $a_k \geq 1$ , for every  $k = 1, 2, \dots, n$ . Hence,  $a_{k+1} \geq a_k + 1$  for  $k = 1, 2, \dots, n - 1$ ; or  $a_1 \geq 1, a_2 \geq 2, \dots, a_n \geq n$ .

Therefore,

$$2021 = a_1 + a_2 + \cdots + a_n \geq 1 + 2 + \cdots + n = \frac{(n+1)n}{2}$$

or, in other words,  $n^2 + n \leq 4042$ . Solving this inequality gives  $n \leq 63$ .  $\square$

*Solution A3.*

The player who picks second wins. One possible winning strategy is the following.

After the first player makes his pick; the second player ensures that, after his pick, the remaining petals are arranged in two groups of adjacent petals with the same number of petals.

Subsequently, after each move of the second player, he ensures that the petals are arranged in even number of groups, and each group has a matching pair with the same number of petals.  $\square$

*Solution A4.*

The number 133 factors  $7 \times 19$ .

The first eight  $\pmod{7}$  remainders of the  $F$  sequence are

$$1, 1, 2, 3, 5, 1, 6, 0.$$

Note that  $f_8 \equiv 0 \pmod{7}$ . Let's prove that  $f_{8k} \equiv 0 \pmod{7}$  for every  $k \in \mathbb{N}$ . The next two elements in the sequence  $\pmod{7}$  are 6 and 6. Hence, the eight remainders, from 9-th to 16-th are

$$6 \times (1, 1, 2, 3, 5, 1, 6, 0) \pmod{7} = 6, 6, 5, 4, 2, 6, 1, 0.$$

Hence, the 17th and 18th elements are 1 and 1 and the sequence of remainders repeat from this point on.

The first eighteen elements of  $\pmod{19}$  remainders are

$$1, 1, 2, 3, 5, 8, 13, 2, 15, 17, 13, 11, 5, 16, 2, 18, 1, 0.$$

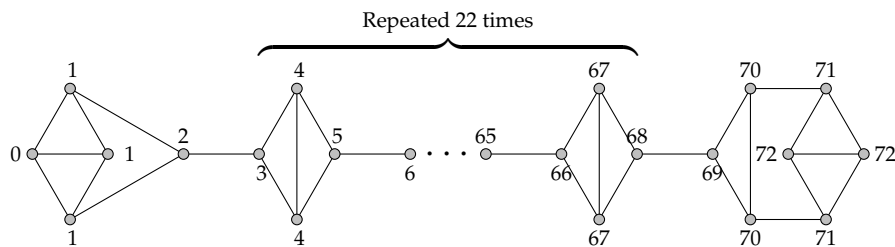
The next two elements are 1 and 1 and the sequence of remainders repeats.

Hence,  $f_{18k} \equiv 0 \pmod{19}$  for every  $k \in \mathbb{N}$ . Since  $\text{lcm}(8, 18) = 72$ , we have  $f_{72k} \equiv 0 \pmod{19 \times 7}$  for every  $k \in \mathbb{N}$ .  $\square$

*Solution A5.*

Answer: Yes.

The acquaintances graph is shown below. Each dot represents a person and each edge represents acquaintance relation. Each integer is the day after 1 Jan ('1' is 1 Jan) the person gets infected. The integer which corresponds to 5 Mar is '64' and the integer which corresponds to 14 Mar is '73'.



$\square$

## B Senior Division – Problems

*Solution B1.*

Let  $\vec{h}_1, \vec{h}_2, \dots$  be the vectors representing the horizontal sections of the ant's path and  $\vec{v}_1, \vec{v}_2, \dots$  be the vectors representing vertical sections of the path. Assume for simplicity that the ant starts in a horizontal direction. We then have

$$\vec{0} = \vec{h}_1 + \vec{v}_1 + \vec{h}_2 + \vec{v}_2 + \dots$$

Since addition is commutative, i.e., we can change the order of terms, we note that

$$\vec{0} = (\vec{h}_1 + \vec{h}_2 + \dots) + (\vec{v}_1 + \vec{v}_2 + \dots) = \vec{H} + \vec{V}.$$

The vector  $\vec{H}$  represents all of the horizontal components and  $\vec{V}$  represents all of the vertical components. Hence,  $\vec{H} + \vec{V} = \vec{0}$ , so  $\vec{H} = \vec{V} = \vec{0}$ .

Let  $k$  be the total number of horizontal components and  $m$  be the total number of vertical components. That is,

$$\vec{H} = \vec{h}_1 + \dots + \vec{h}_k \quad \text{and} \quad \vec{V} = \vec{v}_1 + \dots + \vec{v}_m.$$

Since horizontal and vertical components alternate, we have  $k = m$  or  $k = m \pm 1$ . On the other hand, the ant crawls with constant velocity, so both numbers  $m$  and  $k$  are even. Therefore,  $m = k = 2s$  for some  $s \in \mathbb{N}$  or  $m + k = 4s$ .

Hence, the total travel time is  $15 \times (m + k) = 60 \times (m + k)$ ; or  $(m + k)$ -hours.  $\square$

*Solution B2.*

The first eighteen elements of  $\pmod{19}$  remainders are

$$1, 1, 2, 3, 5, 8, 13, 2, 15, 17, 13, 11, 5, 16, 2, 18, 1, 0.$$

The next two elements are 1 and 1 and the sequence of remainders repeats. Since we have  $2021 = 18 \times 112 + 5$ , the remainder of  $f_{2021}$  will be the 5th in the sequence of remainders above. Hence,  $f_{2021} \equiv 5 \pmod{19}$ .  $\square$

*Solution B3.*

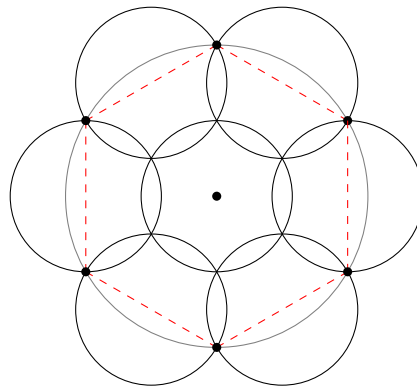
Answer: 7 helicopters.

Within 10mins time, each helicopter is capable of reaching within a disc of radius

$$(300\text{km/h}) \times 10\text{min} = 50\text{km}.$$

Hence, we need to find the minimum number of discs of radius 50 needed to cover the disc of radius 100.

The picture below shows one method of doing so with 7 discs (=helicopters).



Let's prove that the larger disc cannot be covered with 6 discs. One small disc covers at most  $1/6$  of the circumference of the larger disc. Hence, at least six discs are needed to cover the circumference. Also, one smaller disc cannot cover a piece of circumference and the centre at the same time. Hence, at least one extra (seventh) disc is needed to cover the centre point.  $\square$

*Solution B4.*

$$\begin{aligned}4n^4 + m^4 &= (\sqrt{2}n)^4 + m^4 \\ &= (2n^2)^2 + (m^2)^2 + 4n^2m^2 - 4n^2m^2 \\ &= (2n^2 + m^2)^2 - (2nm)^2 \\ &= (2n^2 + m^2 + 2nm)(2n^2 + m^2 - 2nm).\end{aligned}$$

We need to confirm that both factors in the RHS are greater than 1. For the first factor, we note that  $2n^2 + m^2 + 2nm \geq 5$ . For the second factor, assuming  $n \neq m$ , we have  $2n^2 + m^2 - 2nm = n^2 + (n - m)^2 \geq 2$ .  $\square$

*Solution B5.*

We solve this problem by working backward. Assume that the director, the managers and the supervisors are all dismissed. We call this scenario *A*. In the scenario *A*, the each field worker ensures a payout of

$$\$100k/1000 = \$100.$$

Scenario *B*: in this scenario, the director and the managers are dismissed and the supervisors are to propose the plan. By offering a single field worker a payout of \$100, the supervisors secure  $100 \times 10 + 1 = 1001$  votes, and each supervisors receives a payout of

$$(\$100k - \$100)/100 = \$999.$$

Scenario *C*: in this scenario, the director is dismissed and the senior managers are to propose the plan. According to scenario *A* one vote of a field worker is "worth" \$100; and, according to scenario *B*, one vote of a supervisor is "worth"  $\$999/10 = \$99$ . Hence, by offering the first 50 supervisors each a payout of \$999 and by offering a single field worker a payout of \$100, the managers secure  $10 \times 100 + 50 \times 10 + 1 = 1501$  vote, and each managers receives a payout of

$$(\$100k - 50 \times \$999 - \$100)/10 = \$4995.$$

From director's perspective, according to scenario *C*, one vote of a manager costs  $\$4995/10 = \$499.5$ . Hence, to ensure the majority votes for his plan, the director has to offer 10 managers the payout of \$4995 each and one worker a payout of \$100. Such plan secures  $1 \times 1000 + 10 \times 100 + 1 = 2001$  votes, and director receives

$$\$100k - 10 \times \$4995 - \$100 = \$49950.$$

$\square$